Name of College – S. S. College, J. Bad

Deptt – Mathematics

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DATE:

TOPIC – Integration of Some Special Trigonometric functions (Indefinite Integrals)

Class – B.Sc I (Hons)

Time – 10.15 A.M to 11.00 A.M

11.00 A.M to 11.45 A.M.

Date – 04-09-2020

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Form – (i) $\int \frac{dx}{a+b\cos x}$ (ii) $\int \frac{dx}{a+b\sin x}$

(1) Integrate $\int \frac{dx}{5+4\cos x}$

$$= \int \frac{dx}{5(\cos^2\frac{x}{2}+\sin^2\frac{x}{2})+4(\cos^2\frac{x}{2}-\sin^2\frac{x}{2})}$$

$$= \int \frac{dx}{9\cos^2\frac{x}{2}+\sin^2 x/2}$$

$$= \int \frac{dx}{\cos^2 x/2\,(9+\tan^2 x/2)}$$

$$= \int \frac{\sec^2\frac{x}{2}}{\tan^2 x/2+9}\,dx$$

$$= \int \frac{\sec^2 x/2\; dx}{(\tan x/2)^2+3^2}$$

put $\tan\frac{x}{2} = z$

$\frac{1}{2}\sec^2 x/2\, dx = dz$

$\Rightarrow \sec^2 x/2\, dx = 2\,dz$

$3^2 = K^2$

$$= 2\int \frac{dz}{z^2+K^2}$$

$$= 2\cdot\frac{1}{K}\tan^{-1}\frac{z}{K}$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan x/2}{3}\right)+C$$

(2) Integrate $\int \frac{dx}{5+12\cos x}$

$$= \int \frac{dx}{5\left(\cos^2\frac{x}{2}+\sin^2 x/2\right)+12\left(\cos^2 x/2-\sin^2 x/2\right)}.$$

$$= \int \frac{dx}{17\cos^2 x/2 - 7\sin^2 x/2}$$

$$= \int \frac{dx}{\cos^2 x/2\left(17-7\tan^2 x/2\right)}$$

$$= \int \frac{\sec^2 x/2 \, dx}{7\left(\frac{17}{7}-\tan^2 x/2\right)}$$

$$= \frac{1}{7}\int \frac{\sec^2 x/2}{\frac{17}{7}-(\tan x/2)^2}\,dx$$

put $\tan x/2 = z \Rightarrow \frac{1}{2}\sec^2 x/2 \, dx = dz$

$\Rightarrow \sec^2 x/2 \, dx = 2dz$

$$= \frac{1\times 2}{7}\int \frac{dz}{K^2 - z^2}$$

$$\int \frac{dz}{a^2-z^2} = \frac{1}{2}\log\frac{a+z}{a-z}$$

$$= \frac{2}{7}\frac{1}{2K}\log\frac{K+z}{K-z}$$

$$= \frac{2}{7}\times\frac{1}{2K}\log\frac{K+z}{K-z}$$

put $K^2 = \frac{17}{7}$, $K = \sqrt{\frac{17}{7}}$

$z^2 = \tan^2 x/2$, $z = \tan x/2$

$$= \frac{2}{7}\times\frac{1}{2\sqrt{\frac{17}{7}}}\log\frac{\sqrt{\frac{17}{7}}+\tan\frac{x}{2}}{\sqrt{\frac{17}{7}}-\tan x/2} + C_1$$

(3) Integrate

$$\int \frac{dx}{4+5\sin x}$$

$$I = \int \frac{dx}{4+5\sin x}$$

$$= \int \frac{dx}{4(\cos^2 x/2 + \sin^2 x/2) + 5\times 2\sin x/2 \cos x/2}$$

$$= \int \frac{dx}{\cos^2 x/2\left[4 + 4\tan^2 x/2 + 10\tan x/2\right]}$$

$$= \int \frac{\sec^2 x/2 \, dx}{4 + 4\tan^2 x/2 + 10\tan x/2}$$

Put $\tan x/2 = z$

$$\frac{1}{2}\sec^2 x/2 \, dx = dz$$

$$\sec^2 x/2 \, dx = 2\,dz$$

$$= \int \frac{2\,dz}{4+4z^2+10z}$$

$$= \int \frac{2\,dz}{2\cdot 4\left(1+z^2+\frac{10}{4}z\right)}$$

$$= \frac{1}{2}\int \frac{dz}{z^2+\frac{5}{2}z+1}$$

$$= \frac{1}{2}\int \frac{dz}{\left(z+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$$

$$\frac{1}{\cancel{2}\cdot\cancel{2}\cdot\frac{3}{4}}\left\{\log\frac{z+\frac{5}{4}-\frac{3}{4}}{z+\frac{5}{4}+\frac{3}{4}}\right\}$$

$$=\frac{1}{3}\cdot\frac{1}{2}\log\frac{2z+1}{2z+8/4}$$

$$\frac{1}{3}\log\frac{2\tan x/_2+1}{2\tan x/_2+1}+C$$

(4) $I=\int\frac{dx}{\sin x(3+2\cos x)}$

Put $\cos x=z$

$-\sin x\,dx=dz$

$\Rightarrow\sin x\,dx=-dz$

Therefore $I=\int\frac{\sin x\,dx}{\sin^2x(3+2\cos x)}$

$$=\int\frac{dz}{(1-z^2)(3+2z)}$$

$$=-\int\frac{dz}{(1+z)(1-z)(3+2z)}$$

Let $\frac{1}{(1+z)(1-z)(3+2z)}=\frac{A}{1+z}+\frac{B}{1-z}+\frac{C}{3+2z}$

$$1=A(1-z)(3+2z)+B(1+z)(3+2z)+C(1+z)(1-z)$$

Put $z=1$

$1=B(2)(5)\Rightarrow B=\frac{1}{10}$

Put $x = -1$

$$1 = A(1+1)(3-2)$$

$$A = \frac{1}{2}$$

Put $-3/2$

$$(1-3/2)(1+3/2)C = 1$$

$$= -\frac{1}{2} \times \frac{5}{2} \quad -\frac{1}{2} \times \frac{5}{2} C = 1$$

$$= -\frac{5}{4} \quad \therefore C = -4/5$$

$$~~I = -\frac{1}{2}\int \frac{dz}{1+z} \quad \frac{4}{5}\int \frac{dz}{1-z} +~~$$

$$I = -\frac{1}{2}\int \frac{dz}{1+z} - \frac{1}{10}\int \frac{dz}{1-z} + \frac{4}{5}\int \frac{dz}{(3+2z)}$$

$$= -\frac{1}{2}\log(1+z) + \frac{1}{10}\log(1-z) + \frac{4}{5} \times \frac{1}{2}\log(3+2z) + C$$

$$= -\frac{1}{2}\log(1+z) + \frac{1}{10}\log(1-z) + \frac{2}{5}\log(3+2z) + C$$

$$= -\frac{1}{2}\log(1+\cos x) + \frac{1}{10}\log(1-\cos x) + \frac{2}{5}\log(3+2\cos x) + C$$

* $I = \int \sqrt{\sec x - 1}\, dx$

$$= \int \frac{\sqrt{1-\cos x}}{\sqrt{\cos x}}\, dx$$

$$= \int \frac{\sqrt{2}\sin x/2}{\sqrt{2\cos^2 x/2 - 1}}\, dx = \int \frac{\sin\frac{x}{2}\, dx}{\sqrt{\cos^2 x/2 - \frac{1}{2}}}$$

$$= \sqrt{2}\int \frac{\sin x/2 \, dx}{\sqrt{2\cos^2 x/2 - 1}}$$

$$= \frac{\sqrt{2}}{\sqrt{2}}\int \frac{\sin \frac{x}{2} \, dx}{\sqrt{\cos^2 x/2 - \frac{1}{2}}}$$

$$= \int \frac{\sin x/2}{\sqrt{\cos^2 x/2 - 1/2}} \, dx$$

Put $\cos x/2 = z$

$-\frac{1}{2}\sin x/2 \, dx = dz$

$\sin x/2 \, dx = -2dz$

$$= -\int \frac{2 \cdot dz}{\sqrt{z^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -2\int \frac{dz}{\sqrt{z^2 - k^2}} \quad \text{where } k^2 = \frac{1}{2}$$

$$= -2 \log\left[\frac{z + \sqrt{z^2 - k^2}}{k}\right]$$

$$= -2 \log \sqrt{2}\left[\cos x/2 + \sqrt{\cos^2 \frac{x}{2} - \frac{1}{2}}\right]$$

$$= -2\log\left[\cos x/2 + \frac{\sqrt{2\cos^2 x/2 - 1}}{\sqrt{2}}\right] + C$$

$$= -2\log\sqrt{2}\left[\sqrt{2}\cos x/2 + \sqrt{\cos x}\right] + C$$

Integrate

$$\int \sqrt{\sec x + 1}\, dx$$

$$= \int \sqrt{\frac{1+\cos x}{\cos x}}\, dx$$

$$= \int \frac{\sqrt{2\cos^2 x/2}}{\sqrt{1-2\sin^2 x/2}}\, dx$$

$$= \sqrt{2} \int \frac{\cos x/2 \, dx}{\sqrt{2\left(\frac{1}{2} - \sin^2 x/2\right)}}$$

$$= \int \frac{\cos x/2 \, dx}{\sqrt{\frac{1}{2} - \sin^2 x/2}}$$

put $\sin x/2 = z$

$\frac{1}{2} \cos x/2 \, dx = dz$

$\cos x/2 \, dx = 2dz$

$$= 2\int \frac{dz}{\sqrt{K^2 - z^2}}$$

$K^2 = \frac{1}{2}$

$$= 2\sin^{-1}\frac{z}{K} + C$$

$$= 2\sin^{-1}\frac{\sin\frac{x}{2}}{1/\sqrt{2}} + C$$

$$= 2\sin^{-1}(\sqrt{2}\sin x/2) + C$$

Integrate $I = \int \sqrt{\tan\theta}\, d\theta$

Put $\tan\theta = z^2 \Rightarrow \sqrt{\tan\theta} = z.$

$$\sec^2\theta \, d\theta = 2z\, dz$$

$$d\theta = \frac{2z\, dz}{\sec^2\theta}$$

$$= \frac{2z}{1+z^4}\, dz.$$

$$I = \int \frac{z \cdot 2z\, dz}{1+z^4}$$

$$= 2\int \frac{z^2}{1+z^4}\, dz$$

$$= \int \frac{z^2+1 + z^2-1}{z^4+1}\, dz$$

$$= \int \frac{z^2+1}{z^4+1}\, dz + \int \frac{z^2-1}{z^4+1}\, dz$$

$$= \int \frac{\cancel{z^2}\left(1+\frac{1}{z^2}\right)}{\cancel{z^2}\left(z^2+\frac{1}{z^2}\right)}\, dz + \int \frac{\cancel{z^2}\left(1-\frac{1}{z^2}\right)}{\cancel{z^2}\left(z^2+\frac{1}{z^2}\right)}$$

$$= \int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z-\frac{1}{z}\right)^2+2}\, dz + \int \frac{1-\frac{1}{z^2}}{\left(z+\frac{1}{z}\right)^2-2}\, dz$$

Put $z - \frac{1}{z} = t$ Also $z + \frac{1}{z} = p$

$\left(1 + \frac{1}{z^2}\right) dz = dt$ $\left(1 - \frac{1}{z^2}\right) dz = dp$

$$\therefore I = \int \frac{dt}{t^2+2} + \int \frac{dp}{p^2-2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log \frac{p-2}{p+2} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{\left(z - \frac{1}{z}\right)}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log \frac{z + \frac{1}{z} - 2}{z + \frac{1}{z} + 2} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{z^2-1}{\sqrt{2}z}\right) + \frac{1}{2\sqrt{2}} \log \left[\frac{z^2 - 2z + 1}{z^2 + 2z + 1}\right] + C$$